Rutgers Math 492: Math in the Making

Instructors: Dr. Alex Kontorovich and Dr. Glen Whitney

Spring 2023

This semester of the Junior-Senior Honors Seminar is devoted to the interface of mathematics, art, and design. The seminar meets on Wednesdays, 10:20-11:40 am in Hill 525.

Dr. Kontorovich has office hours on Tuesdays 1-2 pm in Hill 630.

Dr. Whitney (glen.whitney@rutgers.edu) has office hours on Wednesdays 2-3 pm in Hill 538.

Both are available for further help by appointment, which could be in-person or over zoom. The zoom link for any course business is:
`https://rutgers.zoom.us/j/94045555690?pwd=alpLR0pVVXVHSlpiSWhMZ0t5WEJVdz09`

1 Introduction

This course is devoted to the study of mathematics via the construction of mathematical objects. "Construction" here refers to faithful digital modeling in software, followed whenever possible either by physical construction by hand or fabrication by 3D printing, laser cutting, CNC cutting, or other digitally-controlled technologies. (Sometimes automated fabrication is used to produce parts that are then hand-assembled.) The goal is to learn interesting mathematics while also learning about design, implementation, and artistry. Rutgers has a great Makerspace, of which we will try to make extensive use.

Each group of N students (with $N = 1$ allowed) will do $N \cdot M$ projects chosen from the list below. (M is determined by number of weeks versus enrollment. The expectation is that $M = 1$ or 2.) Each project consists of:

- one paper (roughly 2-5 pages) describing the mathematics + design/development process, issues + outcomes
- the construction (physical/digital object)
- a 30 minute presentation. (Once we get going, a typical 80 minute session will be comprised of two 30 min presentations with 10 mins of discussion after each.)

If working in a team, each individual is responsible for "leading" M presentations, and for writing M papers.

For any presentation: a software model of the construction is due 1 week before the presentation. A draft of the paper is due at time of presentation. The final version of paper is due one week after presentation. You are expected to show physical models of your construction at your presentation whenever possible.

Recommended software:

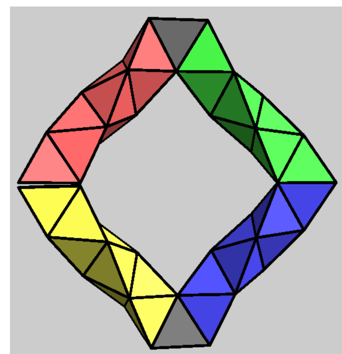
- GeoGebra
- Blender
- OpenSCAD
- LibreCAD

Possible alternatives:

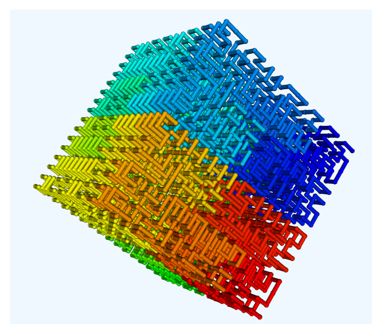
- Rhino (paid, but perhaps accessible through maker space?)
- Mathematica / Sage
- FreeCAD

2 Potential Projects List

1) Rhombic Hexecontahedron of Dodecahedra. 182 regular dodecahedra can be arranged at the vertices and edge midpoints of a rhombic hexecontahedron so that they meet face to face. Any line segment in the resulting solid corresponds to a billiard path in the interior of a dodecahedron. The full configuration, a portion of it, and/or billiard paths in a dodcahedron could be constructed. Suggested software: Blender; suggested construction: 3D printing or solid dodecahedra glued face-to-face.

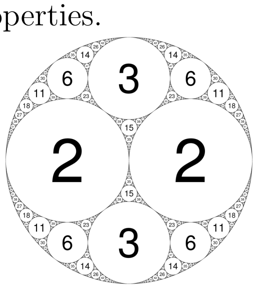


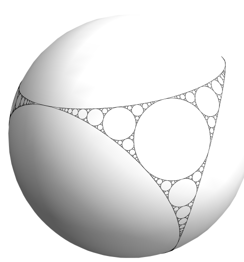
2) Tetrahedral almost loops. It is possible to create tetrahedral paths that come arbitrarily close to closing into loops. Build examples of near misses. Blender; 3D printing or connecting physical tetrahedra; Dr. Whitney has a custom construction system ideal for a large-scale, possibly outdoor, version of this. Reference: Elgersma, M., & Wagon, S. (2016). The quadrahelix: a nearly perfect loop of tetrahedra. arXiv:1610.00280.



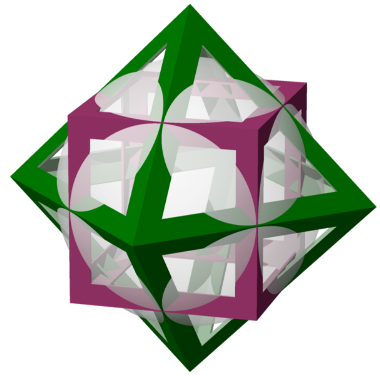
3) Space-filling curves. How does one create a curve that visits *every* point in space? Model could be a sequence of successive approximations to such a curve. OpenSCAD; 3D printing.

4) Apollonian gaskets. Arrangements of circles in a plane or on the surface of a sphere such that each is tangent to three others. Surprisingly, they display deep number-theoretic properties.

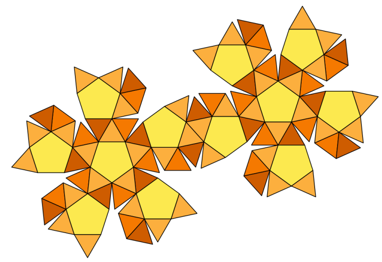




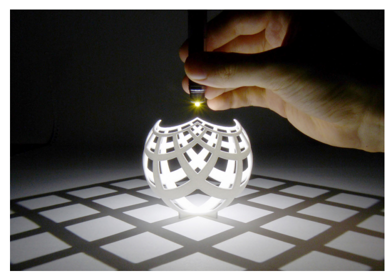
5) Polyhedron-dual compounds. A polyhedron and its dual can be superimposed on a common “midsphere” producing two families of mutually perpendicular spheres. Geogebra; 3D printing or large-scale construction with hula hoops and PVC junctions. Dr. Kontorovich has a useful reference for this project.



6) Polyhedra via nets. Create foldable models of various polyhedra chosen to illustrate a mathematical property. Candidates include but are not limited to: monohedral polyhedra (perhaps monohedral tori), Szilassi polyhedron, non-convex ”bellows” polyhedra, permutohedra, the eight convex deltahedra, “optimal” polyhedra. The results of this project will be included in semester-end installation. Geogebra/Librecad, CNC cutting.

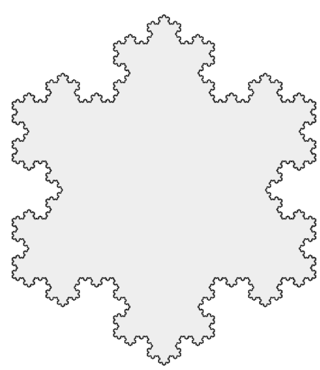


7) Projection shadows. Create a 3D object that produces an interesting shadow when lit at the proper spot. (Note your object is not required to lie in the surface of a sphere, but it may.) OpenSCAD/Mathematica/Sage, 3D printing.

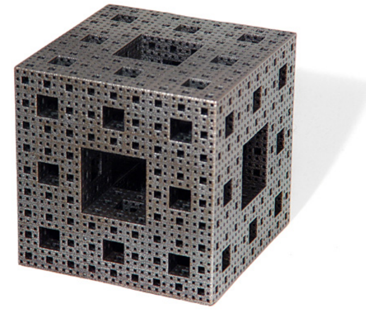


8) Sugihara objects. Create a 3D object with a very different appearance when viewed directly or in a mirror. OpenSCAD/Mathematica/Sage, 3D printing.

9) Koch snowflakes or other 2D fractals. Create successive approximations to a plane figure with infinite perimeter and finite area. GeoGebra/LibreCAD, CNC cutting.

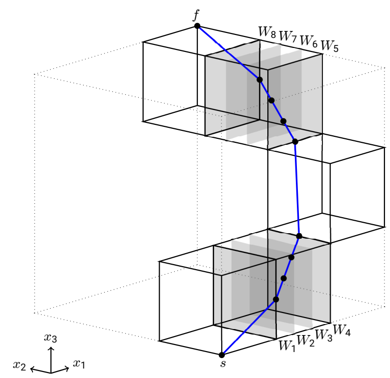


10) Menger sponges or other 3D fractals. Create successive approximations to a solid with infinite surface area and finite (or even zero!) volume. OpenSCAD/Blender, 3D printing.

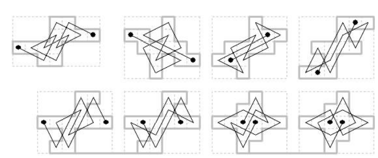


11) Cross sections of Menger Sponge or other 3D fractals: some remarkable/beautiful 2D fractals can be generated in this way. Physical model could be a sliced approximation to the 3D fractal showing the cross section. OpenSCAD/Blender, 3D printing.

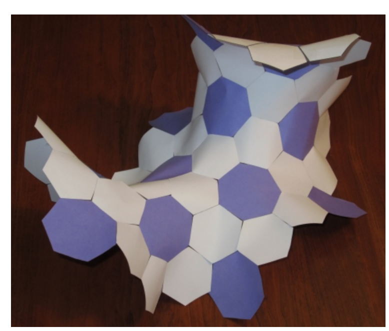
12) Paths in a 3D fractal. Finding the shortest path lying in a Menger sponge that connects diametrically opposite vertices is an open problem. Physical model could be a take-apart Menger sponge revealing a candidate for the shortest path inscribed in the interior. OpenSCAD/Blender, 3D printing. Berkove, E., & Smith, D. (2020). Geodesics in the Sierpinski Carpet and Menger Sponge. *Fractals* Vol. 28, No. 7, 2050120.



13) Knight's tours: When does an edge-connected polyomino admit a knight's tour of all of its cells? Plot/cut resulting tours for interesting regions. Geogebra/LibreCAD, CNC cutting/plotting. See `http://www.mayhematics.com/t/t.htm`



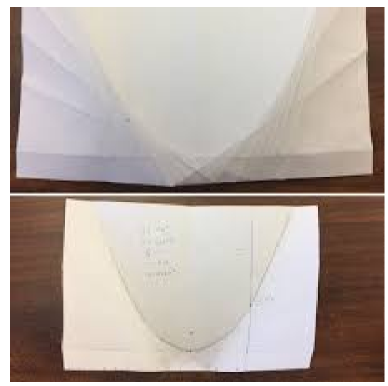
14) Hyperbolic surface model.



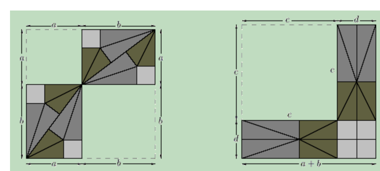
- Take a region of the hyperbolic plane tiled by equilateral tiles, and consider the network of edges.
- Thicken each edge to a strip.
- (Conceptually) cut each vertex in sufficiently many pieces so the network becomes acyclic.
- Lay out the (cut) network onto an ordinary plane and have it cut out.
- Tape all the edges back together at their proper vertices.

You should end up with a model of your original hyperbolic tiling embedded in three dimensions. For some tilings, you can do this with filled polygons rather than just the edge network, maintaining reasonable accuracy. LibreCAD/GeoGebra, CNC cutting and taping.

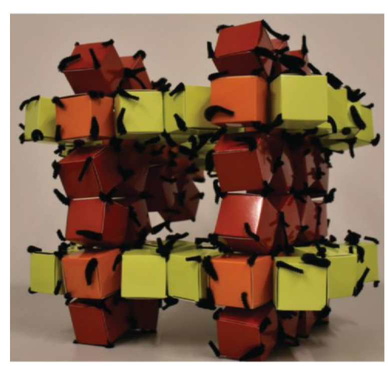
15) Papering the sphere. Starting with a net of a polyhedron and turning its edges into long meanders, one can produce flat cutouts that will wrap a sphere (1-1) with arbitrarily small distortion. Mathematica/Sage, CNC cutting and taping. Delp, K. & Thurston, B. Playing with Surfaces: Spheres, Monkey Pants, and Zippergons. `http://pi.math.cornell.edu/~kdelp/papers/pws.pdf`



16) Origami cubics. Folding schemes for parabolas are well known, as is cubic root-finding by folding. Thus it seems there should be a folding scheme that produces the graph of a cubic. Geogebra/Mathematica/Sage, hand-folding of paper.

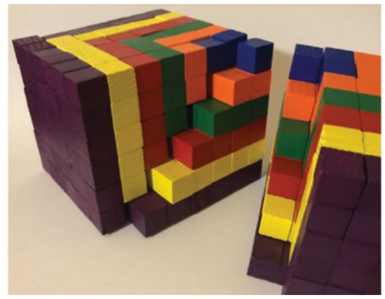


17) Linkage that proves the Pythagorean Theorem: There are remarkably many proofs of the Pythagorean Theorem. Some lend themselves to linkages such that the configuration space of the linkage is in 1-1 correspondence with instances of the pythagorean theorem (up to similarity). Geogebra/LibreCAD, physical construction of the linkage perhaps with support of CNC cutting of the components. Maor, E. & Jost, E. *Beautiful Geometry*, p. 17. See also `https://www.cut-the-knot.org/pythagoras/`.



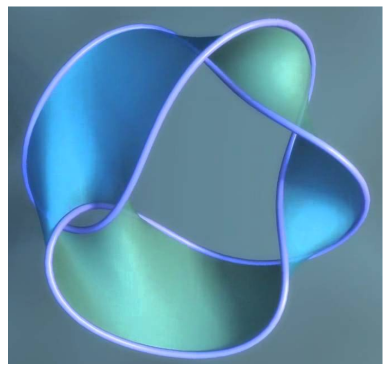
18) Cube linkages. There are many rigid or one-degree-of-freedom (1-DOF) linkages composed of cubes connected flexibly at vertices. You can construct these from colored boxes and pipe cleaners (supplied by the instructors). Examples: a 1-DOF linkage that expands in one dimension when either stretched **or** compressed in a perpendicular direction; the rigid cube-cuboctahedron-rhombicuboctahedron honeycomb, in which remarkably only requires the cubes. Geogebra, hand construction. Bindman, J. et. al. Making Math Material (2019) *Math Horizons* Vol. 27, No. 1, p. 8.

19) Translate a “proof without words” into a physical object. There are both 2D and 3D ones out there that lend themselves to construction. The previous reference contains an example for the sum-of-squares formula. Geogebra/Blender, hand construction or 3D printing.

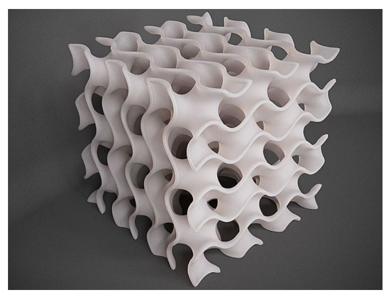


20) Knots. There are numerous interesting physical realizations of knots. Laura Taalman has made models that roll with three points of contact on a planar surface at all times. Or one can create models that achieve various invariants, like the “stick invariant” (minimum number of pieces in a piecewise linear embedding of the knot in 3D). OpenSCAD/GeoGebra, 3D printing.

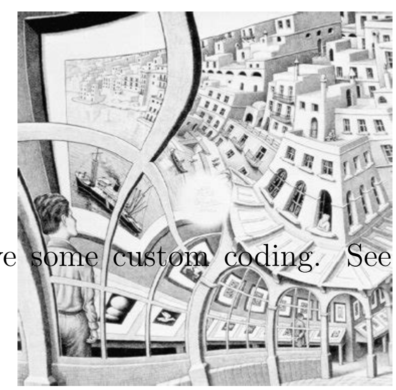
21) Knotted/Seifert surfaces. There are many possibilities here with beautiful forms and deep mathematical connections. Blender/Rhino/Mathematica/Sage, 3D printing.



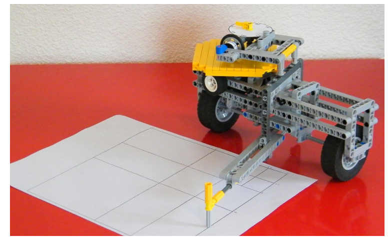
22) Minimal surfaces, like the gyroid. Ditto. Blender / Rhino / Mathematica / Sage, 3D printing.



23) Recreate Escher (continuous) Droste effect. M. C. Escher produced a remarkable drawing in which the scale changes continuously by around a central point, revealing that the subject of the drawing in the bottom left is just a very fine detail of the image that person is viewing on the wall. Can other images be manipulated in the same way? Is it possible to fill in the missing disk in the center? This project might involve some custom coding. See `https://www.ams.org/notices/200304/fea-escher.pdf`.

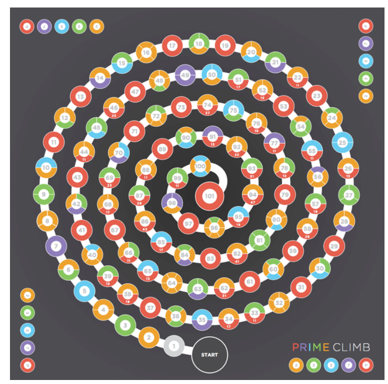


24) Planimeter. This is a device that measures the *area* of a plane region when you trace out its *perimeter* with a stylus. The theory of its operation is (in most cases) based on Green's Theorem. Hand construction, possibly using Lego. See for example `https://www.nico71.fr/orthogonal-planimeter/`.

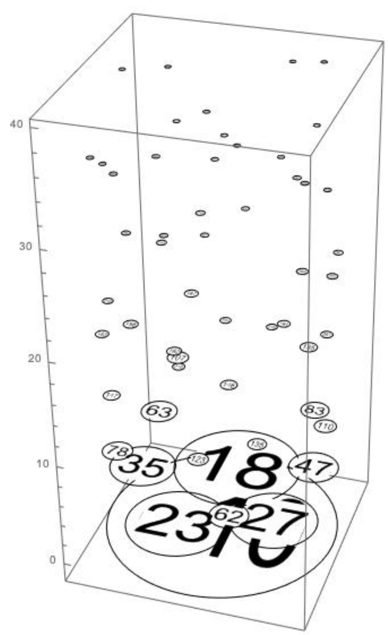


25) Rediscovering Aristarchus. Build a working model of the orbital relationships of the Sun, Moon, and Earth, and use it to explain Aristarchus' calculation of sizes and distances between these heavenly bodies. Dr. Kontorovich has a a seriec of tweets on this topic, see `https://twitter.com/AlexKontorovich/status/1354854112177881092?s=20`.

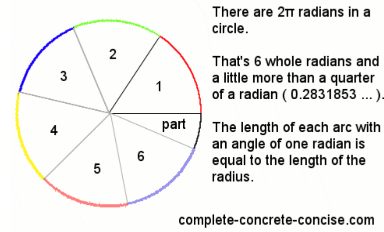
26) Physical factorizations. Build a physical model of the factorizations of the first n natural numbers and use it to illustrate some number-theoretic facts. See for example "Factor City" `https://twitter.com/AlexKontorovich/status/1215657183829995526?s=20`.



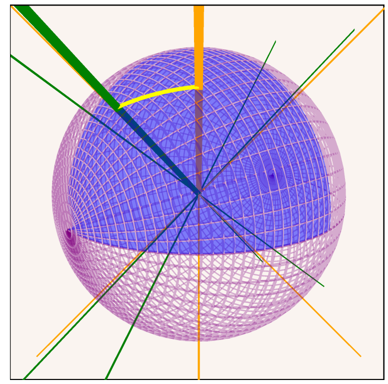
27) Apollonian packing *graded* by curvature. Similar to project 4, but explicitly represent the curvature of each circle, perhaps by the height of a tower or the depth of a pit. OpenSCAD, 3D printing. See `https://twitter.com/AlexKontorovich/status/1229500439315525632?s=20`.



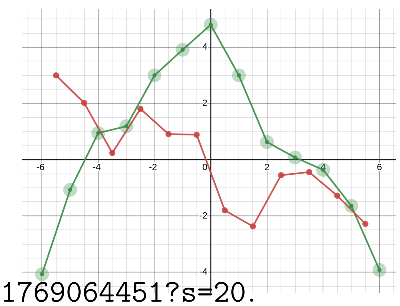
28) Radian "equilateral triangle". Divide a disk completely into 6 slices and an additional partial slice that has 0.2831853 the central angle of the others, together with a mechanism to show that the three edges of the full slices are all identical in (arc) length. LibreCAD, CNC cutting. See `https://twitter.com/AlexKontorovich/status/1227247510785314816?s=20`.



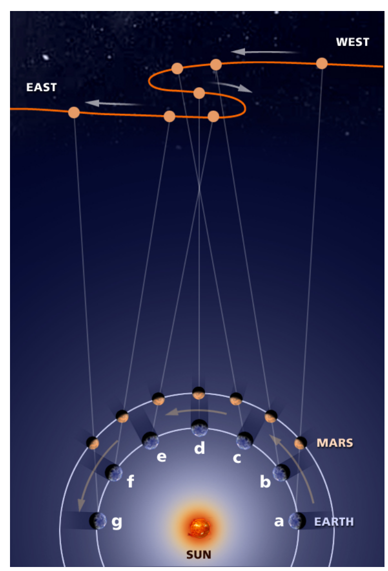
29) A moduli space for triangles. Create a physical model that captures the structure of the space of all triangles, up to similarity. Indicate the points that correspond to special types of triangles, such as acute/right/obtuse, isosceles, etc. What fraction of all triangles are of each type? 3D printing; software depends on the details of your model. See `https://twitter.com/AlexKontorovich/status/1339089375603339265?s=20` and/or `https://sam.zhang.fyi/2019/02/12/a-moduli-space-for-triangles/`



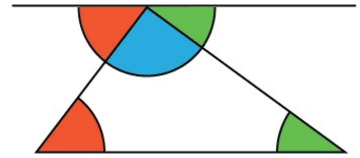
30) Discrete fundamental theorem of calculus mechanisms. Create a mechanism that allows you to set several values of one function at equally-spaced intervals, and it will "display" the approximate values of the derivative and/or definite integral of that function. What settings do you have to put in to get one of the output functions to be a previously specified function? (Like a line or a constant, etc.) See the series of tweets at `https://twitter.com/AlexKontorovich/status/1370194221769064451?s=20`.



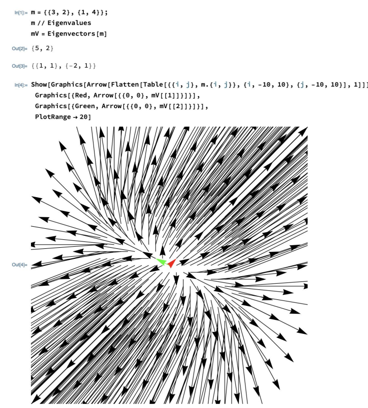
31) Mars retrograde model. Like project 25, but model the Sun-Earth-Mars system with the position of the *Earth* fixed to show why at certain times Mars appears to move "backward" in the sky. See `https://twitter.com/AlexKontorovich/status/1319106271220563968?s=20`.



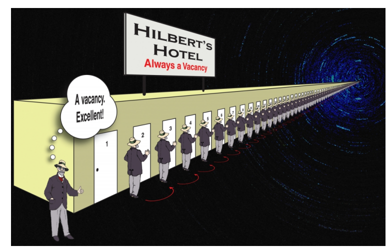
32) Linkage that proves the angles in a triangle add up to a straight angle. (Should work for any triangle.) See `https://twitter.com/AlexKontorovich/status/1442545891110490117?s=20`.



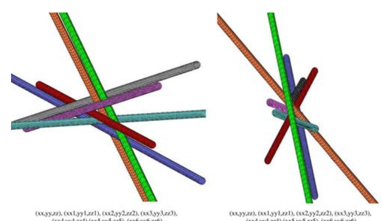
33) Physical demonstration of eigenvalues in linear algebra. See `https://twitter.com/AlexKontorovich/status/1110354039764631553?s=20`.



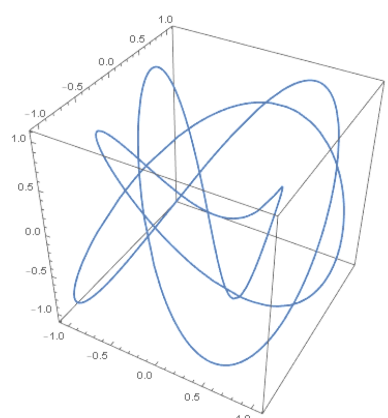
34) Physicalize demonstrations of cardinality. Build a Hilbert Hotel! There are many possible physical illustrations of the bijections used in these arguments; one possibility that could be very attractive is to build a marble run for one of the rearrangements of "guests" at the "hotel". See for example `https://twitter.com/AlexKontorovich/status/1526175079641624576?s=20` or `https://youtu.be/OxGsU8oIWjY`.



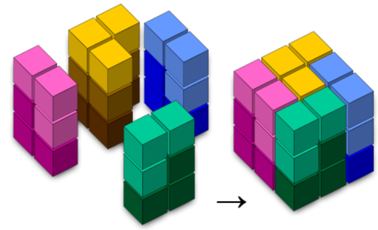
35) Extremal geometric configurations. There are numerous (open and/or solved) problems along the lines of: How many cylinders can each touch all of the others? How many distinct lines can there be containing n out of a fixed set of n^2 points? Construct (candidate) solutions for a chosen problem of this type. See `https://royalsocietypublishing.org/doi/10.1098/rsos.160729` for information on the cylinder problem.



36) 3D generalizations of 2D objects. Choose a classic, well-known 2D curve or region, such as Lissajous curves, dragon curves, or Mandelbrot set. Construct a model of what its 3D analogue should look like.

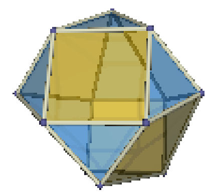


37) Polycube packings. Do the 12 chiral pentacubes and the 12 "solid pentominoes" (pentacubes of height 1) pack into a rectangular solid of volume 120? OpenSCAD/Blender, 3D printing/CNC cutting/hand construction.



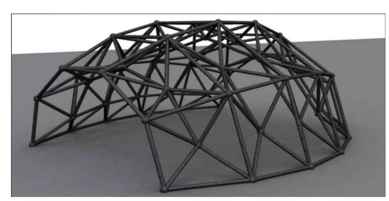
38) Visualizing sequences/3D turtle graphics. Take a familiar sequence, especially one with not many different values such as the ternary square-free sequence, and interpret each value as a distinct direction in three dimensions. Trace out the resulting curve in space. What does it look like, and can you see properties of the sequence in the resulting form? OpenSCAD, 3D printing.

39) Lattice polyhedra. Which regular/archimedean/other class of polyhedra can be realized with vertices being integer lattice points in 3D? Create models of each, or a 3D equivalent of a "geoboard" (pegs and rubber bands) in which each can be constructed. OpenSCAD, 3D printing.



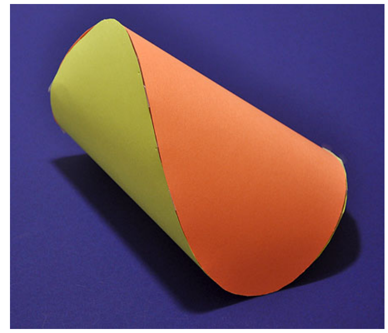
40) Point groups in 2D/3D. Assemble a collection of polygons or polyhedra that illustrate the collection of all point groups in either 2D or 3D. CNC cut or 3D print a gallery of the possibilities. Geogebra/FreeCAD/OpenSCAD, CNC cutting or 3D printing.

41) Rigidity. Construct 3D frameworks (solid struts connected by flexible joints) that are infinitesimally rigid, rigid but not infinitesimally rigid, and have 1 degree of freedom, to compare/contrast. Geogebra, hand-construction.

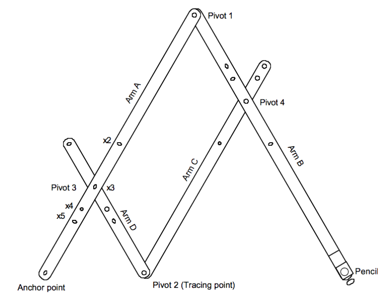


42) Ruled surfaces. Construct models of some interesting ruled/doubly-ruled surfaces, either as "string art" or with rigid rods or as 3D prints. Geogebra/OpenSCAD, hand construction/3D printing.

43) Developable surfaces. Create 2D cutouts that can be assembled into surprising developable surfaces. Examples include tangent developable surfaces, sphericons, D-forms. FreeCAD/LibreCAD/Mathematica/Sage, CNC cutting. See also `http://www.formulas.it/sito/wp-content/uploads/2020/04/pottmann_superfici-sviluppabili.pdf`



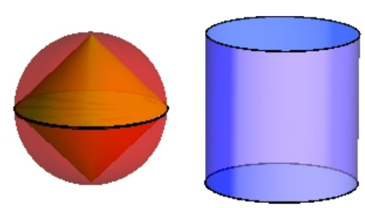
44) Analog “calculators.” For example, there’s an apparatus that computes the geometric mean on p. 27 of “Beautiful Geometry.” by Maor & Jost. GeoGebra, hand-construction (perhaps with the aid of laser-cut or 3D-printed parts).



45) Solids of constant width. There are many interesting ones, including one in a paper by Danzer all of whose cross sections have strictly smaller width. OpenSCAD/Mathematica/Sage, 3D printing. See Kawohl & Weber `http://www.mi.uni-koeln.de/mi/Forschung/Kawohl/kawohl/pub100.pdf` and reference [14] therein (in German).

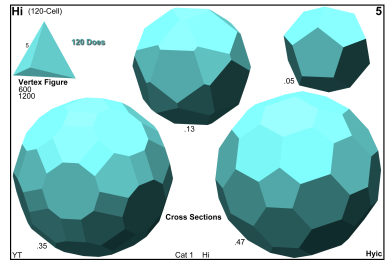
46) Klein bottles. There are numerous interesting immersions in 3D, including minimum-energy ones described in the reference below. OpenSCAD/Mathematica/Sage, 3D printing. Séquin, C. On the number of Klein bottle types (2013). Journal of Mathematical Art.

47) Archimedes’ theorem hourglass. A unit hemisphere plus a unit right circular cone have the same total volume as a unit cylinder. So make one chamber of the hourglass a cylinder minus a cone, say, and the other a hemisphere (or any of various other combinations, possibly using other solids that add up this way for similar reasons) and make a working model where the filling can go from one side to the other to demonstrate the equal volume. OpenSCAD/FreeCAD, 3D printing.

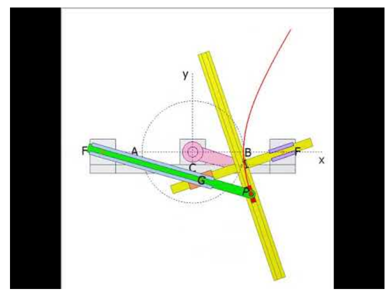


48) Solid of revolution integration demo. Make a set disks and a set of hollow cylinders that each assemble (on some sort of spindle) into the same solid, showing how two different integrals that both express the volume of the same solid of revolution are equal. OpenSCAD/FreeCAD, 3D printing.

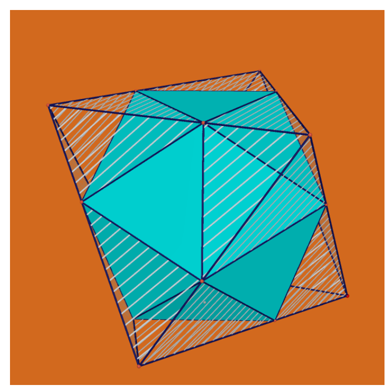
49) Hyperpolyhedra. Create sets of polyhedra that show successive cross-sections of an interesting 4-d polyhedron.



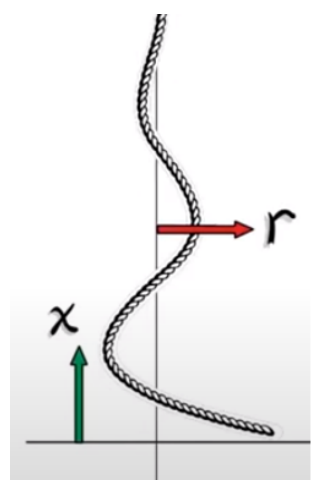
50) Drawing apparatus. Linkages that draw ellipse/parabola/hyperbola/cycloid/etc. Geogebra, hand construction or assembly of 3D printed parts.



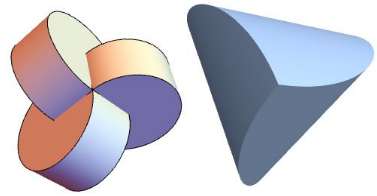
51) Inscribed polyhedra. In each of the regular polyhedra, display the largest-volume copy of each of the other regular polyhedra contained therein. For example, the largest-volume tetrahedron in a cube is the one whose vertices are alternate vertices of the cube, whereas for an icosahedron in an octahedron you obtain the one where eight of the faces of the icosahedron lie in the faces of the octahedron. Geogebra/OpenSCAD, 3D printing.



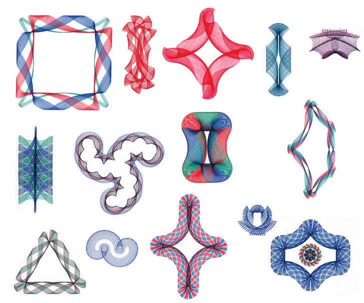
52) Profile of a rotating chain. A chain being "twirled" at its top with the other end free can sweep out a sequence of surfaces, one for each number of nodes in the path of the chain. Determine the corresponding curves for a given length of chain and 3d print the surfaces. OpenSCAD/Mathematica/Sage



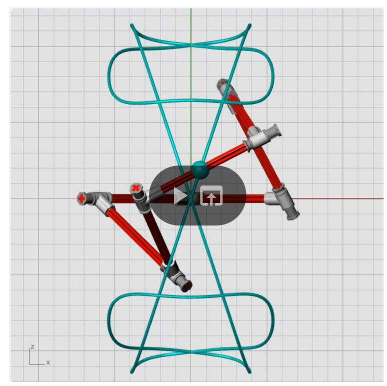
53) Steinmetz solids. Interesting solids consisting of intersections of cylinders. Can be constructed either by direct 3D printing, or by cutting out flat pieces that are rolled onto cylinders (like mailing tubes) and then cut out and glued at the edges to produce the final solids. GeoGebra.



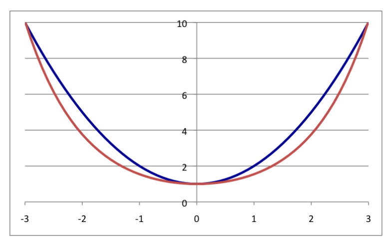
54) Advanced spirograph. Design gears that draw an unusual shape, like a trapezoid, when rolled around inside a circle, or a spirograph where the outer "ring" is a square or a Reuleaux triangle (say). Laser cutting or 3D printing. Variety of possible software models.



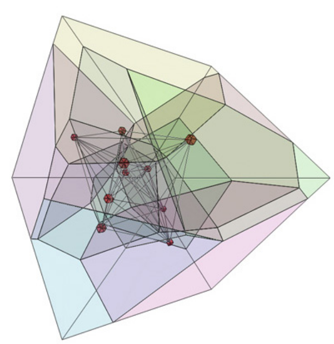
55) A nonplanar regular heptagon with all right angles has one degree of freedom. Build a working model. OpenSCAD/FreeCAD, 3D printing. (Allowing it to actually rotate through all of its configurations will involve devising some sort of swivel joint.) It could also be interesting to produce models of the path a point on the heptagon traces out when one edge is kept rigid while the others go through all of their possible configurations. See `https://twitter.com/KangarooPhysics/status/1446486560250212359`.



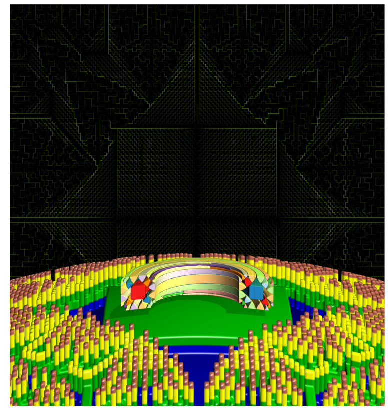
56) Chain graphs. If you hang a chain with no external load, it hangs as a catenary. If you hang equal loads equally spaced horizontally, it hangs as a parabola. Can other familiar curves be obtained by other weight distributions? Build the resulting configurations. Hand construction; may require specialized modeling software or somewhat more extensive Mathematica/Sage coding to simulate the weighted chain.



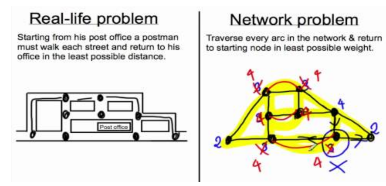
57) Pomegranate cells/3D Voronoi regions. Choose some points in a simple solid (e.g. a cube or sphere) in some way (randomly/via a repelling force/distributed by golden ratio) and print the resulting Voronoi regions in a way that they can be assembled and disassembled (e.g, with sockets where neodymium magnets can be glued). OpenSCAD/Mathematica/Sage, 3D printing.



58) Abelian sandpile models. Could print snapshots of a sandpile process, or more ambitiously a mechanism that would (for at least a small range of iterations) physically execute the sandpile. Blender/OpenSCAD/FreeCAD, 3D printing.

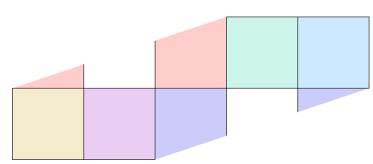


59) Minimal complete traversal of polyhedra. Apply the Route Inspection Problem (RIP) to the network of edges of a polyhedron to find the shortest single piece of string that you can use to string together the polyhedron from tubes; build the resulting models in a way that emphasizes the repeated edges. A similar, but distinct, problem is to find the minimum number of cycles whose union is the given network (i.e., the smallest-cardinality cycle cover SCCC); to illustrate this, you could use a different color for each cycle when stringing the polyhedron to show an optimal solution. Astonishingly, RIP is solvable in polynomial time but SCCC is NP-hard. Geogebra and custom network code implementing the Route Inspection algorithm, hand-construction. A source for references concerning SCCC can be found at `http://users.ece.northwestern.edu/~nickle/pubs/cyclecover.pdf`; algorithms for RIP are easy to find.

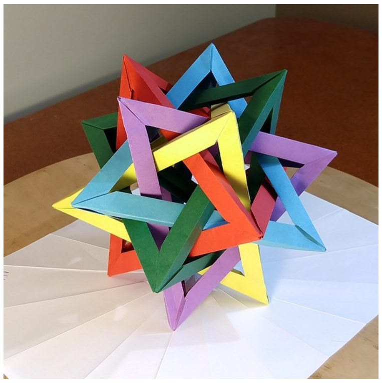


60) Traveling Sales Problem. Create a TSP solver and use it on interesting sets of points in 2D or 3D to produce curves to plot or construct. This will likely involve custom-coded software, with CNC plotting or 3D printing. A key investigator in this area is Bill Cook.

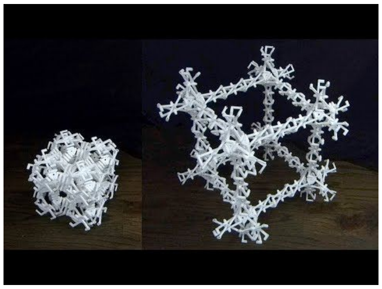
61) Optimal (cube) wrapping. Cut out shapes with scores that illustrate optimal ways of wrapping a cube (or other polyhedron): the smallest square that can wrap a cube, the least-area triangle that can wrap a cube, the dissection of a $\sqrt{6}$ square into the fewest pieces that can wrap a cube, etc. Geogebra, CNC cutting. Catalano-Johnson, M. L. & Loeb, D. "Problem 10716: A cubical gift," American Mathematical Monthly, volume 108, number 1, January 2001, pages 81-82



62) Famous polyhedral compounds. Choose one, such as the compound of five tetrahedra created by partitioning the vertices of a regular dodecahedron. Consider replacing the edges of this compound by solid cylinders (and deleting the faces and interiors of the tetrahedra). There is a maximum radius at which the cylinders do not intersect. At this radius, the structure becomes rigid. 3D print the corresponding models (and perhaps also at slightly less than this radius, showing the non-rigidity in that case). Geogebra/Blender, 3D printing.



63) Expanding polyhedral linkage. Create a mechanism that will allow a polyhedral model either to expand to many times its initial size, or to morph to a related polyhedron (see "jitterbug transformation"), or both simultaneously. Inspired by the "Hoberman Sphere" and/or Henry Segerman's "Extensor Construction Kit". (`http://www.mathmechs.com/`)



64) Roads and wheels. A square wheel will roll on a sequence of catenaries. Just building a 2d model of that is nice, perhaps with gear teeth so that it is sure to roll without slipping. But Stan Wagon in a Math Horizons interview (V6N3, 1999 Feb) states there is a unique road-wheel pair with identical shapes (whatever that means?). Track down the article and the reference and build the resulting road and wheel. FreeCAD/Mathematica/Sage, CNC cutting.